Quantum Harish-Chandra bimodules at roots of unity

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Harish-Chandra $U(\mathfrak{g})$-bimodules

Enveloping algebra $U(\mathfrak{g})$, simply connected algebraic group G.

Definition

Harish-Chandra $U(\mathfrak{g})$-bimodules are $U(\mathfrak{g})$-bimodules having rational adjoint $\mathfrak{g}$-actions , equivalently, (weakly) G-equivariant $U(\mathfrak{g})$-modules.

Typical objects are $V \otimes U(\mathfrak{g})$ for some G-representation V.

Harish-Chandra $U(\mathfrak{g})$-bimodules are related to

- Infinite dimensional representations of complex Lie groups as real groups.
- Category O (Bernstein-Gelfand equivalence)
- Character sheaves (Bezrukavnikov-Finkelberg-Ostrik)
- ...

Harish-Chandra bimodules are defined for filtered quantizations, for the affine Lie algebras, ...

Goal

Define Harish-Chandra bimodules for quantum groups, connect them to affine Soergel bimodules and related objects.

Quantum groups

Quantum group $U_q(\mathfrak{g})$ is the Hopf $\mathbb{C}(q)$-algebra, a deformation of $U(\mathfrak{g})$.

Example

$U_q(\mathfrak{sl}_2)$ is generated by $\{E, F, K, K^{-1}\}$ subject to relations

$KK^{-1} = K^{-1}K = 1,\ \ KE = q^2 EK,\ \ KF = q^{-2}FK,\ \ [E, F] = (K - K^{-1})/(q - q^{-1}).$

- Lusztig form: $U_q^L(\mathfrak{sl}_2)$ is the Hopf $\mathbb{C}[q^{\pm}]$-subalgebra generated by

$$\left\{ E^{(n)} := \frac{E^n}{[n]_q!}, \quad F^{(n)} := \frac{F^n}{[n]_q!}, \quad K^{\pm} \right\}$$

- De Concini-Kac form: $U_q^{DK}(\mathfrak{sl}_2)$ is the Hopf $\mathbb{C}[q^{\pm}]$-subalgebra generated by $\{E, F, K^{\pm}\}$

$$U_q^L(\mathfrak{g}) \leftrightarrow G \qquad\qquad U_q^{DK}(\mathfrak{g}) \leftrightarrow U(\mathfrak{g})$$

Quantum HC bimodules

$\mathrm{Rep}(G)$ = rational reps of G $\rightsquigarrow$ $\mathrm{Rep}(G_q)$ = rational reps of Lusztig form $U_q^L(\mathfrak{g})$

- $\mathrm{Rep}(G_q)$ is braided, i.e, $\sigma_{V,W} : V \otimes W \to W \otimes V$.

Definition

$O_q[G] \cong \bigsqcup_{X \in \mathrm{Rep}(G_q)} X^* \otimes X / \sim$,spanned by matrix coeff of reps in $\mathrm{Rep}(G_q)$.

- $\mathrm{Rep}(G)$ $\rightsquigarrow$ Peter-Weyl theorem $\mathbb{C}[G] \cong \bigoplus_{V \in \mathrm{Irr}(G)} V^* \otimes V$.
- The algebra structure on $O_q[G]$:

$$(V^* \otimes V) \otimes (W^* \otimes W) \xrightarrow{\sigma_{V^* \otimes V, W^*}} W^* \otimes V^* \otimes V \otimes W \cong (V \otimes W)^* \otimes V \otimes W$$

$\Rightarrow$ $O_q[G]$ becomes an algebra in $\mathrm{Rep}(G_q)$.

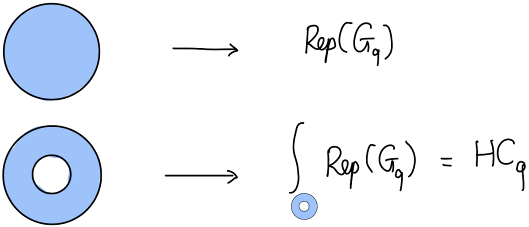
Definition of HC_q

Quantum HC bimodules are right $O_q[G]$-modules in $\mathrm{Rep}(G_q)$.

Example: $V \otimes O_q[G]$ belongs to HC_q for any $V \in \mathrm{Rep}(G_q)$.

Quantum HC bimodules

The categories HC_q showed up in works about TQFT of surfaces via factorization homology (D. Ben-Zvi, A. Brochier, D. Jordan)



Remark

There is an embedding of $U_q^L(\mathfrak{g})$-module algebras $O_q[G] \hookrightarrow U_q^{ev}(\mathfrak{g})$, a modification of $U_q^{DK}(\mathfrak{g})$. Moreover, $O_q[G] \xrightarrow{\sim} U_q^{fin}$, the $U_q^L(\mathfrak{g})$-locally finite part of $U_q^{ev}(\mathfrak{g})$.

HC_q: Monoidal structure

- By concatenating annuli
- By an algebraic way:
  - A Hopf algebra H is a H-module algebra via the left adjoint action
  $$\mathrm{ad}_l(h)(g) = \sum h_{(1)} g S(h_{(2)})$$
  Define $H\text{-Rmod}^H = \{\text{right } H\text{-modules in } \mathrm{Rep}(H)\}$.
  - Any $M \in H\text{-Rmod}^H$ is a H-bimodule:
  $$hm = \sum (h_{(1)} \cdot m) h_{(2)}$$
  - For any $V \in \mathrm{Rep}(G_q)$, then $V \otimes U_q^{ev}$ is a U_q^{ev}-bimodule.

  $\Rightarrow$ $V \otimes O_q[G]$ is a $O_q[G]$-bimodule in $\mathrm{Rep}(G_q)$.

  $\Rightarrow$ Any object in HC_q is $O_q[G]$-bimodule in $\mathrm{Rep}(G_q)$.

Both monoidal structures should be related.

Some results on $O_\epsilon[G]$

Assume $q = \epsilon$ a root of unity of order ℓ (odd). Let Z be the center of $O_\epsilon[G]$

Theorem (Losev, Tsymbaliuk, V.)

$$Z \cong \mathcal{O}(G \times_{T/W} T/W)$$

- $G \to G//G \cong T/W$ and $T/W \to T/W : \quad t \mapsto t^\ell$
- $\mathrm{Fr}^* : \mathrm{Rep}(G) \to \mathrm{Rep}(G_\epsilon) \rightsquigarrow O[G] \hookrightarrow O_\epsilon[G], \qquad O_\epsilon[G]^{U_\epsilon^L} \cong \mathbb{C}[T/W]$

The component T/W in Z is called Harish-Chandra center $Z_{\epsilon,HC}$ of $O_\epsilon[G]$.

Let $G_0^{reg} = \{\text{regular elements in Bruhat cell in } G\} \subset G$.

Theorem (L-T-V)

$O_\epsilon[G]$ is a finitely generated module over Z, Azumaya over $G_0^{reg} \times_{T/W} T/W$.

Principal block $\mathrm{HC}_\epsilon(0,0)$

Any object in HC_ϵ is a $O_\epsilon[G]$-bimodule. For any $M \in \mathrm{HC}_\epsilon$,

$$Z_{\epsilon,HC} \curvearrowright M \curvearrowleft Z_{\epsilon,HC}$$

The counit $O_\epsilon[G] \to \mathbb{C}$ gives the trivial character of $Z_{\epsilon,HC}$. Let $Z_{\epsilon,HC}^{\wedge_0}$ be the completion of $Z_{\epsilon,HC}$ at this trivial character.

Definition

$\mathrm{HC}_\epsilon(0,0)$ is the completed version of HC_ϵ where the left and right action of $Z_{\epsilon,HC}$ on any object of $\mathrm{HC}_\epsilon(0,0)$ can be extended to actions of $Z_{\epsilon,HC}^{\wedge_0}$.

Example

$O_\epsilon^\wedge[G] := O_\epsilon[G] \otimes_{Z_{\epsilon,HC}} Z_{\epsilon,HC}^{\wedge_0}$.

Affine Soergel bimodules

$W_{aff} := W \ltimes Q \curvearrowright \mathbb{C}\hbar \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan subspace of $\mathfrak{g}$.

$$w(a,x) = (a, wx), \quad t_\lambda(a,x) = (a + \langle \lambda, x \rangle, x), \quad \lambda \in Q, x \in \mathfrak{h}, w \in W, a \in \mathbb{C}.$$

$\mathrm{R} := S[[\mathbb{C}\hbar \oplus \mathfrak{h}]]$, a completion of the symmetric power.
R-bimod : the category of R-bimodules.

Example

Bott-Samelson bimodule $\mathrm{R} \otimes_{\mathrm{R}^s} \mathrm{R}$ for a simple reflection $s \in W_{aff}$.

Definition

$\mathrm{SB}_\hbar$ is the full Karoubian monoidal subcategory of the category R-bimod generated by Bott-Samelson bimodules BS_s.

Definition

SB has same objects as $\mathrm{SB}_\hbar$ but the morphisms are defined by

$$\mathrm{Hom}_{\mathrm{SB}}(M,N) := \mathrm{Hom}_{\mathrm{SB}_\hbar}(M,N)/\hbar\mathrm{Hom}_{\mathrm{SB}_\hbar}(M,N).$$

Affine Soergel bimodules

The category SB and $\mathrm{SB}_\hbar$ are categorification of $\mathbb{Z}[W_{aff}]$, i.e,

$$\begin{array}{rccc} & K_0(\mathrm{SB}) & \cong & \mathbb{Z}[W_{aff}] \\ \text{indecomposable object} & [B_x] & \leftrightarrow & C_x \quad \text{Kazhdan-Lusztig basis} \end{array}$$

• Two-sided cells: $y \overset{LR}{\sim} x$ for $x, y \in W_{aff}$ iff

$$C_h C_x C_{h'} = aC_y + \dots, \qquad C_k C_y C_{k'} = bC_x + \dots$$

for some nonzero $a, b \in \mathbb{Z}$.

• $W_{aff} = \{\text{two-sided cells}\}$. There is the smallest two-sided cell under a certain order.

This smallest two-sided cell corresponds to a full subcategory $\mathcal{C} \subset \mathrm{SB}$.

Main results

Let ϵ a root of unity of order ℓ (odd + some mild restrictions)
Let us assume the weights of objects in $\mathsf{HC}_\epsilon(0,0)$ are contained in the root lattice.
Let $\mathsf{Proj}_\epsilon(0,0) = \{$projective objects in $\mathsf{HC}_\epsilon(0,0)\}$.

Theorem (V.)

- There is an embedding of additive monoidal categories

$$\mathsf{SB} \hookrightarrow \mathsf{HC}_\epsilon(0,0),$$

which restricts to an equivalence

$$\mathcal{C} \cong \mathsf{Proj}_\epsilon(0,0).$$

- There is an equivalence of triangulated categories

$$K^b(\mathsf{SB}) \cong D^b(\mathsf{HC}_\epsilon(0,0)).$$

Compare to Soergel results

$V(\nu)$: the irreducible G-rep with the highest weight contained in the orbit $W\nu$.

Z : the center of $U(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{h}^*/(W,\bullet)]$.
$Z^{\wedge_0}$: the completion of Z at the trivial central character then

$$Z^{\wedge_0} \cong \mathbb{C}[[\mathfrak{h}^*]] := S.$$

$U^0 := U(\mathfrak{g}) \otimes_Z Z^{\wedge_0}$.
$\mathrm{HC}(0,0) :=$ category of U^0-bimodules having rational adjoint $\mathfrak{g}$-actions.

Weyl group $W \curvearrowright \mathfrak{h}^*$.
$\mathrm{BS}_s := S \otimes_{S^s} S$ for simple reflection $s \in W$
$\rightsquigarrow$ The category of finite Soergel bimodules SB^{fin}.

Theorem (Soergel 92)

There is a full embedding of monoidal categories

$$\mathbb{V} : \mathrm{SB}^{fin} \hookrightarrow \mathrm{HC}(0,0)$$

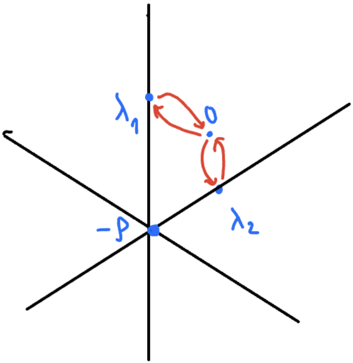
Compare to Soergel results

The category O : $(\mathfrak{g}, B)$-modules, contains the Verma modules.

The principal block $O_{[0]}$ containing the trivial representation.

$$\mathrm{HC}(0,0) \curvearrowright O_{[0]}$$

Choose a ρ-dominant weight λ_i in the hyperplane of $\mathfrak{h}^*_{\mathbb{R}}$ defined by simple reflection $s_i \in W$. Consider irreducible reprensentations $V(\lambda_i)$ and $V(-\lambda_i)$.



We obtain translation functors:

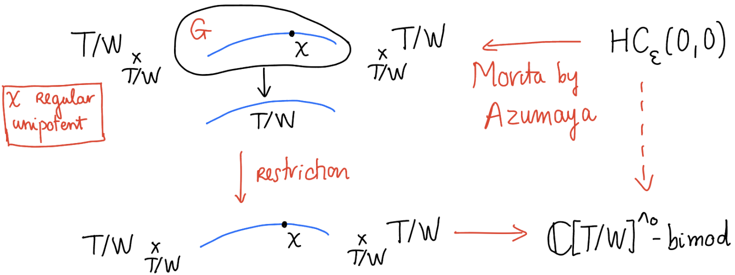
$$O_{[0]} \to O_{[\lambda_i]} \qquad M \mapsto \mathrm{pr}_{[\lambda_i]}(V(\lambda_i) \otimes M)$$
$$O_{[\lambda_i]} \to O_{[0]} \qquad N \mapsto \mathrm{pr}_{[0]}(V(-\lambda_i) \otimes N),$$

$\rightsquigarrow P^{\lambda_i,0} \in \mathrm{HC}(\lambda_i, 0)$ and $P^{0,\lambda_i} \in \mathrm{HC}(0, \lambda_i)$.
We define

$$\Theta_i := P^{0,\lambda_i} \otimes_{U^{\lambda_i}} P^{\lambda_i,0}.$$

Then $\mathbb{V}(\mathrm{BS}_{s_i}) = \Theta_i$

- Any object in HC_ϵ is a module over $\mathcal{O}_\epsilon[G] \otimes_{\mathcal{O}[G]} \mathcal{O}_\epsilon^{op}[G]$
- The center of $\mathcal{O}_\epsilon[G] \otimes_{\mathcal{O}[G]} \mathcal{O}_\epsilon^{op}[G]$ is $\mathcal{O}(T/W \times_{T/W} G \times_{T/W} T/W)$.
- Recall that $O_\epsilon[G]$ is Azumaya over $G_0^{reg} \times_{T/W} T/W$



- $\exists$ Poisson structure on Z. Let deform $\epsilon \in \mathbb{C} \mapsto q = \epsilon e^{\hbar} \in \mathbb{C}[[\hbar]]$:

$$Z_{\epsilon,HC}^{\wedge_0} \rightsquigarrow Z_{q,HC}^{\wedge_0} = \mathbb{C}[[\hbar]][T/W]^{\wedge_0} \cong \mathrm{R}, \qquad \mathrm{HC}_\epsilon(0,0) \rightsquigarrow \mathrm{HC}_q(0,0)$$

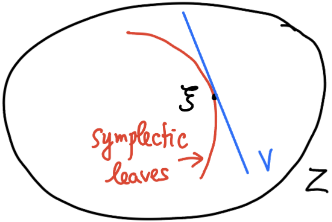
So we have the functor

$$\bullet_\dagger : \mathrm{HC}_q(0,0) \to Z_{q,HC}^{\wedge_0}\text{-bimod} \cong \mathrm{R}\text{-bimod} \supset \mathrm{SB}_\hbar$$

The functor $\bullet_\dagger$: construction

$\xi := (\chi, 0) \in G \times_{T/W} T/W$ with a regular unipotent χ in G_0 and the trivial character 0 of $Z_{\epsilon,HC}$. Let $\mathfrak{m}_\xi$ be the maximal idea in Z

$\phi : O_q[G] \xrightarrow{/\hbar} O_\epsilon[G]$. Let $J_\xi := \phi^{-1}(O_\epsilon[G]\mathfrak{m}_\xi)$.



Let $O_\epsilon[G]^{\wedge\xi}$ and $O_q[G]^{\wedge\xi}$ be the completions of the algebras at $O_\epsilon[G]\mathfrak{m}_\xi$ and J_ξ.

$$O_\epsilon[G]^{\wedge\xi} \cong \mathrm{Mat}_d(\mathbb{C}) \otimes \mathbb{C}[[V]]\widehat{\otimes} Z_{\epsilon,HC}^{\wedge 0},$$
$$O_q[G]^{\wedge\xi} \cong \mathrm{Mat}_d(\mathbb{C}) \otimes \mathcal{A}_\hbar \widehat{\otimes} Z_{q,HC}^{\wedge 0},$$

Here $\mathcal{A}_\hbar$ is the complete formal Weyl algebra, e.g,

$$\mathbb{C}\langle\langle \hbar, x, y\rangle\rangle/(xy - yx = \hbar)$$

The functor $\bullet_\dagger$: construction

$$O_q[G]^{\wedge_\xi} \cong \mathrm{Mat}_d(\mathbb{C}) \otimes \mathcal{A}_\hbar \widehat{\otimes} Z_{q,HC}^{\wedge_0}$$

- For any $M \in \mathrm{HC}_q(0,0)$, the completion $M^\wedge := \varprojlim M/MJ_\xi^k$ is a bimodule over $O_q[G]^{\wedge_\xi}$ equipped with Poisson structures.
- Let $e := E_{11} \in \mathrm{Mat}_d(\mathbb{C})$ then $eM^\wedge e$ is a bimodule over $\mathcal{A}_\hbar \widehat{\otimes} Z_{q,HC}^{\wedge_0}$ with Poisson structures.
- The Poisson structure allows to decompose

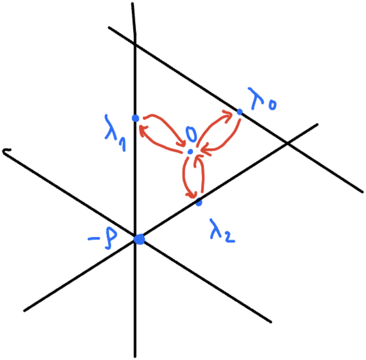
$$eM^\wedge e \cong \mathcal{A}_\hbar \widehat{\otimes} M_\dagger$$

so that $M_\dagger$ is a bimodule over $Z_{q,\mathrm{HC}}^{\wedge_0}$.

Hence the functor $\bullet_\dagger$ is constructed by

$$M \mapsto M^\wedge \mapsto eM^\wedge e \mapsto M_\dagger.$$

The embedding $\mathrm{SB} \hookrightarrow \mathrm{HC}_\epsilon(0,0)$



For any simple reflection s_i of W_{aff}, there is a bimodule $\Theta_{\hbar,i}$ such that

$$(\Theta_{\hbar,i})_\dagger \cong \mathrm{BS}_{s_i}$$

By proving a fully faithfulness property of $\bullet_\dagger$, we obtain an embedding

$$\mathrm{SB}_\hbar \hookrightarrow \mathrm{HC}_q(0,0).$$

This induces the desired embedding

$$\mathrm{SB} \hookrightarrow \mathrm{HC}_\epsilon(0,0).$$

Equivalence $\mathcal{C} \cong \mathrm{Proj}_\epsilon(0,0)$

- Let w_0 be the longest element in W, then $\mathcal{C}$ contains the indecomposable Soergel bimodule B_{w_0}. Indecomposable objects in $\mathcal{C}$ are direct summands of

$$P_1 \star B_{w_0} \star P_2,$$

  for some $P_1, P_2 \in \mathrm{SB}$.
- Construct projective object Θ such that B_{w_0} is sent to Θ.
- The category $\mathrm{HC}_\epsilon(0,0)$ is Krull-Schmidt so that we can talk about projective covers of simple objects.
- Classify simple objects in $\mathrm{HC}_\epsilon(0,0)$. Prove that for any simple L_1, L_2 then L_1 is a subquotient of $P_1 \star L_2 \star P_2$ for some $P_1, P_2 \in \mathrm{SB}$. Use this to show that $\mathrm{Proj}_\epsilon(0,0)$ can be recovered from Θ and the action of SB as in the first bullet point.

Non-commutative Springer resolution

Let $\tilde{\mathfrak{g}} \to \mathfrak{g}$ be the Grothendieck-Springer resolution.

Let $\mathrm{St} := \tilde{\mathfrak{g}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ be the Steinberg variety

There is a vector bundle $\mathcal{E}$ on $\tilde{\mathfrak{g}}$ so that we have an equivalence, $\mathsf{A} = \mathrm{End}(\mathcal{E})^{op}$,

$$D^b(\mathrm{Coh}(\tilde{\mathfrak{g}})) \cong D^b(\mathsf{A}\text{-mod}), \qquad \mathcal{F} \mapsto \mathrm{RHom}(\mathcal{E}, \mathcal{F}).$$

So A is called the Non-commutative Springer resolution.

The last part is included in the following result

Theorem (V.)

There are equivalences of triangulated categories:

$$K^b(\mathrm{SB}) \cong D^b(\mathrm{HC}_\epsilon(0,0)) \cong D^b((\mathsf{A} \otimes_{\mathbb{C}[\mathfrak{g}]} \mathsf{A}^{op})^\wedge\text{-mod}^G) \cong D^b\mathrm{Coh}^G(\mathrm{St}^\wedge),$$

in which the middle equivalence is t-exact with the standard t-structures.

Remarks

- Study other blocks of quantum Harish-Chandra bimodules
- Study its relation with quantum category O, quantum character sheaves, ...
- In positive characteristics,

Conjecture

When characteristic p is good and ϵ has a large enough odd order, there is an equivalence of triangulated categories:

$$D^b(\mathrm{HC}_\epsilon(0,0)) \cong D^b(\mathrm{Coh}^G(\mathrm{St}_m^\wedge)).$$

Here St_m is the Steinberg variety for the unipotent cone of the algebraic group G.

Thank you!